MATH20132 Calculus of Several Variables. 2020-21

Solutions to Problems 2: Continuity

The definition of continuity given in the notes *is that* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is continuous at* $\mathbf{a} \in U$ *if, and only if,* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$. *This has the* ε - δ *version*

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \forall \mathbf{x},\ |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \varepsilon.$$

1. Scalar-valued functions.

i. Let $1 \le i \le n$ and define the *i-th projection function* $p^i : \mathbb{R}^n \to \mathbb{R}$ by only retaining the i-th coordinate, so

$$p^i(\mathbf{x}) = p^i\left(\left(x^1, ..., x^n\right)^T\right) = x^i.$$

Verify the ε - δ definition to show that p^i is continuous on $\mathbb{R}^n$.

Remember, if $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$ *and* $|\mathbf{x} - \mathbf{a}| < \delta$ *then* $|x^i - a^i| < \delta$ *for each* $1 \le i \le n$.

A different proof of continuity was given in the lectures.

ii. Prove, by verifying the ε - δ definition that

$$f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto x^1 + x^2 + ... + x^n$$

is continuous on $\mathbb{R}^n$.

iii. Let $\mathbf{c} \in \mathbb{R}^n, \mathbf{c} \ne \mathbf{0}$, be a fixed vector. Prove, by verifying the ε - δ definition that $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ is continuous on $\mathbb{R}^n$.

Hint Make use of the Cauchy-Schwarz inequality, $|\mathbf{c} \bullet \mathbf{d}| \le |\mathbf{c}|\,|\mathbf{d}|$ for $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$.

Part i is a special case of Part iii, with $\mathbf{c} = \mathbf{e}_i$, *while Part ii is the special case* $\mathbf{c} = (1, 1, ..., 1)^T$.

Solution i. Let $1 \le i \le n$ be given. Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon > 0$. Assume $\mathbf{x}$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$. This means in particular that $|x^i - a^i| < \delta$. For such $\mathbf{x}$ consider

$$\left|p^i(\mathbf{x}) - p^i(\mathbf{a})\right| = \left|x^i - a^i\right| < \delta = \varepsilon.$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} p^i(\mathbf{x}) = p^i(\mathbf{a})$. So p^i is continuous at $\mathbf{a}$. But i and $\mathbf{a}$ were arbitrary, so p^i is continuous on $\mathbb{R}^n$ for all i.

ii. Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/n > 0$. Assume $\mathbf{x}$ satisfies $|\mathbf{x}-\mathbf{a}| < \delta$. This means that $|x^i - a^i| < \delta$ for all components. For such $\mathbf{x}$ consider

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{a})| &= \left|\left(x^1 + x^2 + ... + x^n\right) - \left(a^1 + a^2 + ... + a^n\right)\right| \\
&= \left|\left(x^1 - a^1\right) + \left(x^2 - a^2\right) + ... + \left(x^n - a^n\right)\right| \\
&\leq \left|x^1 - a^1\right| + \left|x^2 - a^2\right| + ... + \left|x^n - a^n\right| \\
&\qquad \text{by the triangle inequality} \\
&< n\delta = n\left(\frac{\varepsilon}{n}\right) = \varepsilon.
\end{aligned}$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$. So f is continuous at $\mathbf{a}$. But $\mathbf{a}$ was arbitrary, so f is continuous on $\mathbb{R}^n$.

iii. One solution is to follow part ii. Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon / \sum_{i=1}^n |c^i| > 0$. Assume $\mathbf{x}$ satisfies $|\mathbf{x}-\mathbf{a}| < \delta$. For such $\mathbf{x}$ consider

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{a})| &= |\mathbf{c} \bullet \mathbf{x} - \mathbf{c} \bullet \mathbf{a}| \\
&= \left|\left(c^1x^1 + c^2x^2 + ... + c^nx^n\right) - \left(c^1a^1 + c^2a^2 + ... + c^na^n\right)\right| \\
&= \left|c^1\left(x^1 - a^1\right) + c^2\left(x^2 - a^2\right) + ... + c^n\left(x^n - a^n\right)\right| \\
&\leq \left|c^1\right|\left|x^1 - a^1\right| + \left|c^2\right|\left|x^2 - a^2\right| + ... + \left|c^n\right|\left|x^n - a^n\right| \\
&\qquad \text{by the triangle inequality} \\
&< \delta \sum_{i=1}^n \left|c^i\right| = \sum_{i=1}^n \left|c^i\right| \left(\frac{\varepsilon}{\sum_{i=1}^n \left|c^i\right|}\right) = \varepsilon.
\end{aligned}$$

Hence we have verified the definition of f continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary, f is continuous on $\mathbb{R}^n$.

Alternative solution Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given. Choose

$\delta = \varepsilon / |\mathbf{c}| > 0$. Assume $\mathbf{x}$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$. For such $\mathbf{x}$ consider

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{a})| &= |\mathbf{c} \bullet \mathbf{x} - \mathbf{c} \bullet \mathbf{a}| \\
&= |\mathbf{c} \bullet (\mathbf{x} - \mathbf{a})| \qquad \text{since the scalar product is distributive} \\
&\leq |\mathbf{c}|\, |\mathbf{x} - \mathbf{a}| \qquad \text{by Cauchy-Schwarz} \\
&< |\mathbf{c}|\, \delta \\
&= |\mathbf{c}|\, (\varepsilon / |\mathbf{c}|) \\
&= \varepsilon.
\end{aligned}$$

Hence we have verified the definition of f continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary, f is continuous on $\mathbb{R}^n$.

2 Prove, by verifying the ε - δ definition of continuity that the scalar-valued $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto xy$ is continuous on $\mathbb{R}^2$.

Hint If $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ is given write $f(\mathbf{x}) - f(\mathbf{a}) = xy - ab$ in terms of $x - a$ and $y - b$.

Solution The method is based on the identity

$$xy - ab = (x - a)(y - b) + a(y - b) + b(x - a). \tag{1}$$

Let $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ be given. Let $\varepsilon > 0$ be given. Choose

$$\delta = \min\left(1, \frac{\varepsilon}{1 + |a| + |b|}\right) > 0.$$

Assume $\mathbf{x} = (x, y)^T$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$, in which case

$$|x - a| < \delta \quad \text{and} \quad |y - b| < \delta. \tag{2}$$

For such $\mathbf{x}$ consider

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{a})| &= |xy - ab| = |(x - a)(y - b) + a(y - b) + b(x - a)| \\
&\qquad \text{by (1) above,} \\
&\leq |x - a|\, |y - b| + |a|\, |y - b| + b\, |x - a| \\
&\qquad \text{by the triangle inequality,} \\
&< \delta^2 + |a|\, \delta + |b|\, \delta,
\end{aligned}$$

by (2). We are also assuming that $\delta \leq 1$ in which case $\delta^2 \leq \delta$ and thus

$$\begin{aligned}
|f(\mathbf{x}) - f(\mathbf{a})| &< \delta\,(1 + |a| + |b|) \\
&\leq \frac{\varepsilon}{1 + |a| + |b|}\,(1 + |a| + |b|) \\
&\qquad \text{since } \delta < \varepsilon/\,(1 + |a| + |b|) \\
&= \varepsilon.
\end{aligned}$$

Hence we have verified the definition of f continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary, f is continuous on $\mathbb{R}^2$.

Note You might try to use the identity

$$xy - ab = (x - a)\,y + a\,(y - b)\,.$$

This would lead to

$$|f(\mathbf{x}) - f(\mathbf{a})| \leq \delta|y| + \delta|a|\,.$$

You could NOT choose $\delta = \varepsilon/(|y| + |a|)$, since δ **cannot** depend on the varying point $\mathbf{x} = (x, y)^T$. It can only depend on the fixed point $\mathbf{a} = (a, b)^T$.

Instead you should demand that $\delta \leq 1$ when $|y - b| < \delta \leq 1$ opens out to give $|y| < 1 + |b|$. Then

$$|f(\mathbf{x}) - f(\mathbf{a})| \leq \delta|y| + \delta|a| \leq \delta(1 + |b| + |a|)\,,$$

and we choose the same δ as above.

3 Prove, by verifying the ε - δ definition that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix}$$

is continuous on $\mathbb{R}^2$.

Note For practice I have asked you to verify the definition, **not** to use any result that would allow you to look at each component separately.

Solution i. Let $\mathbf{a} = (a,b)^T \in \mathbb{R}^2$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/\sqrt{17}$. Assume $\mathbf{x} = (x,y)^T$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$. Then,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})|^2 &= \left| \begin{pmatrix} 2x+y \\ x-3y \end{pmatrix} - \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix} \right|^2 \\ &= \left| \begin{pmatrix} 2(x-a)+(y-b) \\ (x-a)-3(y-b) \end{pmatrix} \right|^2 . \end{aligned}$$

I have written this in terms of $x-a$ and $y-b$ since I know I can make them small. Continue, using the definition of $|...|$ on $\mathbb{R}^n$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})|^2 &= \big(2(x-a)+(y-b)\big)^2 + \big((x-a)-3(y-b)\big)^2 \\ &= 4(x-a)^2 + 4(x-a)(y-b) + (y-b)^2 \\ &\qquad + (x-a)^2 - 6(x-a)(y-b) + 9(y-b)^2 \\ &= 5(x-a)^2 - 2(x-a)(y-b) + 10(y-b)^2 . \end{aligned}$$

The negative sign on the middle term is a possible problem when applying upper bounds for $|x-a|$ and $|y-b|$. We remove this by using the triangle inequality:

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})|^2 &= \left|5(x-a)^2 - 2(x-a)(y-b) + 10(y-b)^2\right| \\ &\leq 5(x-a)^2 + 2|x-a||y-b| + 10(y-b)^2 , \end{aligned}$$

Yet $|\mathbf{x} - \mathbf{a}| < \delta$ means that both $|x-a| < \delta$ and $|y-b| < \delta$. Thus

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})|^2 \leq 5\delta^2 + 2\delta^2 + 10\delta^2 = 17\delta^2.$$

Taking roots gives

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| \leq \sqrt{17}\delta = \sqrt{17}\left(\frac{\varepsilon}{\sqrt{17}}\right) = \varepsilon.$$

Hence $\mathbf{f}$ is continuous at $\mathbf{a} \in \mathbb{R}^2$. Yet $\mathbf{a}$ was arbitrary so $\mathbf{f}$ is continuous on $\mathbb{R}^2$.

Alternative Solution Recall that $|\mathbf{y}| \leq \sum_{i=1}^n y^i$ for $\mathbf{y} \in \mathbb{R}^n$ so $|\mathbf{g}(\mathbf{x})| \leq \sum_{i=1}^m |g^i(\mathbf{x})|$ for any $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$. With $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})$ we get

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| &\leq |2(x-a)+(y-b)| + |(x-a)-3(y-b)| \\ &\leq 2|x-a| + |y-b| + |x-a| + 3|y-b| , \end{aligned}$$

by additional applications of the triangle inequality. Thus

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| \le 3|x-a| + 4|y-b|,$$

and $\delta = \varepsilon/7$ will suffice.

4 Let $M_{m,n}(\mathbb{R})$ be the set of all $m \times n$ matrix of real numbers. Let $M \in M_{m,n}(\mathbb{R})$.

In the notes we showed that the function $\mathbf{x} \mapsto M\mathbf{x}$ is continuous on $\mathbb{R}^n$ by showing that each component function is continuous on $\mathbb{R}^n$. In this question we show it is continuous by verifying the ε - δ definition.

i. Prove that there exists $C > 0$, depending on M, such that $|M\mathbf{x}| \le C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.

   Hint Write the matrix in row form as

$$M = \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ \vdots \\ \mathbf{r}^m \end{pmatrix} \quad \text{when} \quad M\mathbf{x} = \begin{pmatrix} \mathbf{r}^1 \bullet \mathbf{x} \\ \mathbf{r}^2 \bullet \mathbf{x} \\ \vdots \\ \mathbf{r}^m \bullet \mathbf{x} \end{pmatrix}.$$

   What is $|M\mathbf{x}|$? Apply Cauchy-Schwarz to each $|\mathbf{r}^i \bullet \mathbf{x}|$.

ii. Deduce, by verifying the ε - δ definition, that the vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$ is continuous on $\mathbb{R}^n$.

Solution i From the hint given and the definition of the norm we have

$$|M\mathbf{x}|^2 = \sum_{i=1}^m |\mathbf{r}^i \bullet \mathbf{x}|^2 \le \sum_{i=1}^m |\mathbf{r}^i|^2 |\mathbf{x}|^2,$$

by Cauchy-Schwarz. The result then follows with

$$C = \left(\sum_{i=1}^m |\mathbf{r}^i|^2 \right)^{1/2} = \left(\sum_{i=1}^m \sum_{j=1}^n \left(a_j^i\right)^2 \right)^{1/2},$$

where a_j^i is the i, j-th element of M.

ii. Assume $M \neq 0$ since the result is immediate if $M = 0$. Let $\mathbf{f}(\mathbf{x}) = M\mathbf{x}$. Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given, choose $\delta = \varepsilon/C$, with C as found in part a, and $C \neq 0$ since $M \neq 0$. Assume $0 < |\mathbf{x} - \mathbf{a}| < \delta$. Then

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| &= |M\mathbf{x} - M\mathbf{a}| = |M(\mathbf{x} - \mathbf{a})| \\ &\qquad \text{since matrix multiplication is distributive} \\ &\leq C\,|\mathbf{x} - \mathbf{a}| \quad \text{by the definition of } C \\ &< C\delta = C\,(\varepsilon/C) = \varepsilon. \end{aligned}$$

Hence we have verified the definition that $\mathbf{f}$ is continuous at $\mathbf{a} \in \mathbb{R}^n$. Yet $\mathbf{a}$ was arbitrary so $\mathbf{f}$ is continuous on $\mathbb{R}^n$.

5. Determine where each of the following maps $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. For $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$,

i.

$$f(\mathbf{x}) = \begin{cases} x + y & \text{if } y > 0 \\ x - y - 1 & \text{if } y \leq 0 \end{cases}$$

ii.

$$f(\mathbf{x}) = \begin{cases} x + y & \text{if } y > 0 \\ x - y & \text{if } y \leq 0 \end{cases}$$

Hint: Your arguments should split into three cases, $y > 0$, $y < 0$ and $y = 0$. You should make use of the fact that polynomials in x and y are continuous in open subsets of $\mathbb{R}^2$.

Solution i. This function is continuous on the open set given by $y > 0$ (the upper half plane) since it is given by the polynomial $x + y$. It is also continuous on the open set given by $y < 0$ (the lower half plane) because it is given by the polynomial $x + y - 1$.

However, where the upper and lower half plane meet, i.e. the x-axis, f is **not** continuous. This is because, at a point $(x, 0)^T$ on the x-axis we can look at the directional limit as we approach the point on a vertical straight line from *above*, i.e.

$$f\left(\begin{pmatrix} x \\ 0 \end{pmatrix} + t\mathbf{e}_2\right) = f\left(\begin{pmatrix} x \\ t \end{pmatrix}\right) = x + t \to x \quad \text{as} \quad t \to 0+\,.$$

Whereas, approaching the point from *below* on a vertical straight line, the directional limit is

$$f\left(\begin{pmatrix} x \\ 0 \end{pmatrix} + t\mathbf{e}_2\right) = f\left(\begin{pmatrix} x \\ t \end{pmatrix}\right) = x - t - 1 \to x - 1 \quad \text{as} \quad t \to 0- .$$

Different directional limits mean there is no limit at $(x, 0)^T$ and so no continuity there.

ii. This function is continuous since it can be written $f\left((x, y)^T\right) = x + |y|$. (Formally, this is continuous because it is the sum of two continuous functions: $(x, y)^T \mapsto x$ and $(x, y)^T \mapsto y$ are continuous by a result in the lecture notes (and also Question 1 on projections) and $y \mapsto |y|$ is continuous since $\lim_{y\to a} |y| = |a|$ for all $a \in \mathbb{R}$.)

Note This is rather a 'clever' solution of part ii. We could, instead, follow part i and say that this function is continuous on the open set given by $y > 0$ (the upper half plane) since it is given by the polynomial $x + y$. It is also continuous on the open set given by $y < 0$ (the lower half plane) because it is given by the polynomial $x - y$.

Again this leaves the x - axis, but this time we believe that f is continuous there. We show this by verifying the definition of limit. Let $\mathbf{a}$ be an element of the x-axis, so $\mathbf{a} = (a, 0)^T$. Note that $f(\mathbf{a}) = a$. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/2$. Assume $|\mathbf{x} - \mathbf{a}| < \delta$. With $\mathbf{x} = (x, y)^T$ this implies $|x - a| < \delta$ and $|y - 0| < \delta$.

There are two cases, when $y > 0$ and then $y \le 0$.

In the first case, $|\mathbf{x} - \mathbf{a}| < \delta$ and $y > 0$ together give

$$|f(\mathbf{x}) - f(\mathbf{a})| = |(x + y) - a| = |(x - a) + y| \le |x - a| + |y| < 2\delta = \varepsilon,$$

having used the triangle inequality. Similarly in the second case, $|\mathbf{x} - \mathbf{a}| < \delta$ and $y \le 0$ together give

$$|f(\mathbf{x}) - f(\mathbf{a})| = |(x - y) - a| = |(x - a) - y| \le |x - a| + |y| < 2\delta = \varepsilon.$$

In both cases $|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$ and so we have verified the definition of continuity at $\mathbf{a}$. Yet $\mathbf{a}$ was arbitrary so f is continuous on the x-axis.

6. Return to the function of Question 10 Sheet 1, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{(x^2 - y)^2}{x^4 + y^2} \quad \text{for } \mathbf{x} = (x, y)^T \ne \mathbf{0} \quad \text{and} \quad f(\mathbf{0}) = 1.$$

i. Show that f is continuous at the origin along any straight line through the origin.

ii. Show that f is not continuous at the origin.

This is then an illustration of

$$\boxed{\forall \mathbf{v},\ \lim_{t\to 0} f(\mathbf{a}+t\mathbf{v}) = f(\mathbf{a}) \ \not\Longrightarrow\ \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).}$$

Solution i. Continuous at the origin along any straight line through the origin means $\lim_{t\to 0} f(t\mathbf{v}) = f(\mathbf{0})$ for all vectors $\mathbf{v}$. Yet in Question 10i, Sheet 1, you were asked to show that $\lim_{t\to 0} f(t\mathbf{v}) = 1$ and, since $f(\mathbf{0}) = 1$ by the definition of f, we can deduce that f is continuous at the origin along any straight line through the origin.

ii. To be continuous at the origin we require $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = f(\mathbf{0})$. Yet you were required to show in Question 10ii, Sheet 1, that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist. Hence it cannot be continuous at the origin.

Linear Functions.

7. Linear functions The definition of a linear function $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is that

$$\mathbf{L}(\mathbf{u}+\mathbf{v}) = \mathbf{L}(\mathbf{u}) + \mathbf{L}(\mathbf{v}) \quad \text{and} \quad \mathbf{L}(\lambda\mathbf{u}) = \boldsymbol{\lambda}\mathbf{L}(\mathbf{u}),$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.

i. Given $\mathbf{a} \in \mathbb{R}^n$ prove that $L : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{a} \bullet \mathbf{x}$ is a linear function.

   This was stated without proof in the lectures.

ii. An example of Part i is, if $\mathbf{a} = (2, -5)^T \in \mathbb{R}^2$, then $f(\mathbf{x}) = \mathbf{a} \bullet \mathbf{x} = 2x - 5y$ is a linear function on $\mathbb{R}^2$. Show that

   a. $f(\mathbf{x}) = 2x - 5y + 2$ is not a linear function on $\mathbb{R}^2$,

   b. $f(\mathbf{x}) = 2x - 5y + 3xy$ is not a linear function on $\mathbb{R}^2$.

iii. Given $M \in M_{m,n}(\mathbb{R})$, an $m \times n$ matrix with real entries, prove that $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$ is a linear function.

   This was stated without proof in the lectures.

iv. Let $\mathbf{L} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\mathbf{L}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y \\ x - y + 1 \\ 5x \end{pmatrix}.$$

Show that $\mathbf{L}$ is not a linear function.

Solution i. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} L(\mathbf{u} + \mathbf{v}) &= \mathbf{a} \bullet (\mathbf{u} + \mathbf{v}) = \mathbf{a} \bullet \mathbf{u} + \mathbf{a} \bullet \mathbf{v} = L(\mathbf{u}) + L(\mathbf{v}) \\ L(\lambda \mathbf{u}) &= \mathbf{a} \bullet (\lambda \mathbf{u}) = \lambda \mathbf{a} \bullet \mathbf{u} = \boldsymbol{\lambda} L(\mathbf{u}). \end{aligned}$$

Hence L is a linear function.

ii. a. For a counter-example note that

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = -1 \quad \text{and} \quad f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = -4.$$

Since

$$f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) \neq 2f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

we conclude that f is not linear

b. For a counter-example note that

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 0 \quad \text{and} \quad f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = 6.$$

iii. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \mathbf{L}(\mathbf{u}) + \mathbf{L}(\mathbf{v}) &= M\mathbf{u} + M\mathbf{v} = M(\mathbf{u} + \mathbf{v}) = \mathbf{L}(\mathbf{u} + \mathbf{v}) \\ \mathbf{L}(\lambda \mathbf{u}) &= M(\lambda \mathbf{u}) = \lambda M\mathbf{u} = \boldsymbol{\lambda} \mathbf{L}(\mathbf{u}). \end{aligned}$$

Hence $\mathbf{L}$ is a linear function.

iv. For a counter-example note that

$$\mathbf{L}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{L}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 6 \\ 3 \\ 10 \end{pmatrix}.$$

Thus

$$\mathbf{L}\left(2\begin{pmatrix}1\\0\end{pmatrix}\right) \neq 2\mathbf{L}\left(\begin{pmatrix}1\\0\end{pmatrix}\right).$$

Hence $\mathbf{L}$ is not a linear function.

8. If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function prove that there exists $C > 0$, depending on $\mathbf{L}$, such that

$$|\mathbf{L}(\mathbf{x})| \leq C\,|\mathbf{x}| \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Deduce that $\mathbf{L}$ satisfies the ε - δ definition of continuous on $\mathbb{R}^n$.

Hint Apply a result from the lectures along with Question 4 above.

Solution In the notes it is shown that to each linear map is associated a matrix M so that $\mathbf{L}(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The result (3), and the continuity of $\mathbf{L}$, then follows immediately from Question 4 above.

Alternative Solution Given $\mathbf{x} \in \mathbb{R}^n$ we can write $\mathbf{x} = \sum_{i=1}^n x^i \mathbf{e}_i$. Then $\mathbf{L}$ linear means that

$$\mathbf{L}(\mathbf{x}) = \sum_{i=1}^n x^i \mathbf{L}(\mathbf{e}_i).$$

By the triangle inequality,

$$|\mathbf{L}(\mathbf{x})| \leq \sum_{i=1}^n \left|x^i\right| |\mathbf{L}(\mathbf{e}_i)| \leq \left(\sum_{i=1}^n \left|x^i\right|^2\right)^{1/2} \left(\sum_{i=1}^n |\mathbf{L}(\mathbf{e}_i)|^2\right)^{1/2},$$

by Cauchy-Schwarz. This means the required result follows with $C = \left(\sum_{i=1}^n |\mathbf{L}(\mathbf{e}_i)|^2\right)^{1/2}$.

Solutions to Additional Questions 2

9. Verify the ε - δ definition of continuity and show that the scalar-valued $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto x^2 y$ is continuous on $\mathbb{R}^2$.

Hint Given $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ write $x^2y - a^2b$ in terms of $x - a$ and $y - b$.

Solution Let $\mathbf{a} = (a, b)^T \in \mathbb{R}^2$ be given. Let

$$\sigma = \min\left(1, \varepsilon / \left(1 + 3|\mathbf{a}| + 3|\mathbf{a}|^2\right)\right).$$

Assume $\mathbf{x} \in \mathbb{R}^2$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$, so $|x - a| < \delta$ and $|y - b| < \delta$. We start by developing an identity.

$$\begin{aligned} x^2y - a^2b &= (x-a)^2(y-b) + x^2b + 2xay - 2xab - a^2y \\ &= (x-a)^2(y-b) + (x-a)^2 b + 2xay - a^2y - a^2b \\ &= (x-a)^2(y-b) + (x-a)^2 b + 2(x-a)a(y-b) + a^2y \\ &\qquad +2xab - 3a^2b \\ &= (x-a)^2(y-b) + b(x-a)^2 + 2a(x-a)(y-b) \\ &\qquad +a^2(y-b) + 2ab(x-a) \end{aligned}$$

Thus, by the triangle inequality,

$$\begin{aligned} \left|x^2y - a^2b\right| &\leq |x-a|^2|y-b| + |b||x-a|^2 + 2|a||x-a||y-b| \\ &\qquad + |a|^2|y-b| + 2|ab||x-a| \\ &< \delta^3 + |b|\delta + 2|a|\delta^2 + |a|^2\delta + 2|a||b|\delta \\ &< \delta\left(1 + 3|\mathbf{a}| + 3|\mathbf{a}|^2\right), \end{aligned}$$

having used $\delta \leq 1$ and $|a|$, $|b| \leq |\mathbf{a}|$. Then by our choice of δ

$$\left|x^2y - a^2b\right| < \frac{\varepsilon}{1 + 3|\mathbf{a}| + 3|\mathbf{a}|^2}\left(1 + 3|\mathbf{a}| + 3|\mathbf{a}|^2\right) = \varepsilon.$$

And so we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$. Hence f is continuous at $\mathbf{a}$. Yet $\mathbf{a}$ was arbitrary, so f is continuous on $\mathbb{R}^2$.

There is no great virtue in this question other than showing how time consuming it is to verify the definition, even with quite simple functions.

10. Let $1 \leq i \leq n$ and define $\rho^i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by omitting the i-th coordinate, so

$$\rho^i\left(\left(x^1, ..., x^n\right)^T\right) = \left(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n\right)^T.$$

i. Verify the ε-δ definition of continuity and show that ρ^i is continuous on $\mathbb{R}^n$.

ii. For each $1 \leq i \leq n$ find $M_i \in M_{n-1,n}(\mathbb{R})$ such that $\rho^i(\mathbf{x}) = M_i\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. (Thus continuity follows from Question 4. We could, though, note that ρ^i is linear in which case continuity follows from Question 8.)

Solution i. Let $1 \leq i \leq n$, $\mathbf{a} \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$ and assume $\mathbf{x}$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$. Then for such $\mathbf{x}$

$$\begin{aligned}\left|\rho^i(\mathbf{x}) - \rho^i(\mathbf{a})\right|^2 &= \left|\left(x^1 - a^1, ..., x^{i-1} - a^{i-1}, x^{i+1} - a^{i+1}, ..., x^n - a^n\right)^T\right|^2 \\ &= \sum_{j=1, j\neq i}^{n} \left|x^j - a^j\right|^2 \leq \sum_{j=1}^{n} \left|x^j - a^j\right|^2 \\ &= |\mathbf{x} - \mathbf{a}|^2 .\end{aligned}$$

Thus

$$\left|\rho^i(\mathbf{x}) - \rho^i(\mathbf{a})\right| \leq |\mathbf{x} - \mathbf{a}| < \delta = \varepsilon,$$

and we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \rho^i(\mathbf{x}) = \rho^i(\mathbf{a})$. Hence ρ^i is continuous at $\mathbf{a}$. Yet i and $\mathbf{a}$ were arbitrary, so ρ^i is continuous on $\mathbb{R}^2$ for all i.

ii.

$$M_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

with 1's on the two half diagonals, 0's elsewhere, and 0's in the i-th column. The continuity of ρ^i would then also follows from Question 4.